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Analysis of a nonoverlapping domain decomposition method for elliptic partial differential equations[1]

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Abstract

In this study we analyze a nonoverlapping domain decomposition method for the solution of elliptic Partial Differential Equation (PDE) problems. This domain decomposition method involves the solution of Dirichlet and Neumann PDE problems on each subdomain, coupled with smoothing operations on the interfaces of the subdomains. The convergence analysis of the method at the differential equation level is presented. The numerical results confirm the theoretical ones and exhibit the computational efficiency of the method.

Keywords: Domain decomposition; Partial differential equations; Interface relaxation

AMS classification: 35A25; 35J99; 65N99

1. Introduction

The objective of this paper is to analyze an iterative nonoverlapping domain decomposition method for elliptic problems, in which, at odd iteration levels, we exchange Dirichlet boundary values among subdomain problems at their interfaces, while at even iteration levels, we exchange Neumann boundary values.

It seems that although this method has been considered and implemented in previous studies, a rigorous analysis of sharp convergence estimates has not yet given in the literature. In particular in [6] a convergence analysis of the method is carried out at a differential level using Hilbert space

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techniques. In [5] the Galerkin finite element method and the hybrid mixed finite element method are employed to give discrete versions of this method. In this note, we apply Fourier analysis to show the fast convergence rate of this domain decomposition method in case of constant coefficients and two rectangular subdomains. Although real application problems are more complicated, the analysis of the simple case gives us some feel on the general behavior of the method, since any robust method should perform well on this simple example. The sharp convergence results obtained in this paper show that this domain decomposition procedure converges very rapidly, and preconditioning is not necessary (as opposed to some other methods). As in most similar methods, the convergence properties of our scheme depend on a pair of relaxation parameters. However, theoretical analysis and numerical experiments will show that we can simply set them equal to $\frac{1}{2}$ for many important computations.

Several domain decomposition schemes similar to our method have been recently proposed and analyzed (see [4]). In particular, in the one considered by Funaro et al. [1], Neumann values are passed from one subdomain to its neighbor, while Dirichlet values are received from its neighbor. Expected convergence and numerical performance results were obtained. However, the subdomain problems in this method are not parallelizable, since information passing is required between subdomains at the same iteration level. In [2], a convex combination of Neumann and Dirichlet data is passed from each subdomain to its neighbors. This method allows an arbitrary decomposition of the domain and each subdomain problem plays the same role in the computation. However, the convergence of this method is very slow unless some parameter is carefully chosen.

The organization of the rest of this paper is as follows. In Section 2, the domain decomposition method is described for general elliptic problems. In Section 3, the convergence analysis is carried out for rectangular subdomains and constant coefficients. In Section 4, numerical experiments are provided to confirm the theory obtained in Section 3.

2. Formulation of the proposed domain decomposition method

Let Ω be a convex polygon in R^d, $d=1,2,\dots$ with boundary $\partial\Omega$. Consider the following boundary value problem: Given $f\in L^2(\Omega)$ and $g\in H^{\frac{1}{2}}(\partial\Omega)$ find $u\in H^1(\Omega)$ such that

$$Lu=f \quad \text{in } \Omega, \qquad u=g \quad \text{on } \partial\Omega, \tag{1}$$

where the operator L is defined by

$$Lu \equiv -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u. \tag{2}$$

Although the domain decomposition method of this paper is easily formulated and has been implemented with great success for arbitrary decompositions, for simplicity in the analysis that follow, we partition the domain Ω into two nonoverlapping subdomains Ω_1 and Ω_2 such that

$$\bar{\Omega}=\bar{\Omega}_1\cup\bar{\Omega}_2, \qquad \Omega_1\cap\Omega_2=0, \qquad \partial\Omega_1\cap\partial\Omega\neq 0, \qquad \partial\Omega_2\cap\partial\Omega\neq 0. \tag{3}$$

We denote the interface of the two subdomains by $\Gamma=\partial\Omega_1\cap\partial\Omega_2$.

It is well known that, under suitable regularity conditions, the problem (1)–(2) is equivalent to the following split problem:

$$\left.\begin{array}{l} Lu_1 = f \quad \text{in } \Omega_1, \\ u_1 = g \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ u_1 = u_2 \quad \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial v^1} + \dfrac{\partial u_2}{\partial v^2} = 0 \quad \text{on } \Gamma, \end{array}\right| \begin{array}{l} Lu_2 = f \quad \text{in } \Omega_2, \\ u_2 = g \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \\ u_2 = u_1 \quad \text{on } \Gamma, \\ \dfrac{\partial u_2}{\partial v^2} + \dfrac{\partial u_1}{\partial v^1} = 0 \quad \text{on } \Gamma, \end{array} \tag{4}$$

where, for $n = 1, 2$, $u_n = u|_{\Omega_n}$ and where v^n is the outward unit normal vector to $\partial\Omega_n$.

Let us define the following domain decomposition method. Choose $u_n^{(0)} \in H^1(\Omega_n)$ with $u_n^{(0)} 0_n|_{\partial\Omega_n \cap \partial\Omega} = g$, $n = 1, 2$. For $k = 0, 1, 2, \ldots$, we construct the sequence $u_n^{(k+1)} \in H^1(\Omega_n)$ with $u_n^{(k+1)}|_{\partial\Omega \cap \partial\Omega_n} = g$ satisfying:

$$Lu_1^{(2k+1)} = f \quad \text{in } \Omega_1, \qquad u_1^{(2k+1)} = \alpha u_1^{(2k)} + (1-\alpha) u_2^{(2k)} \quad \text{on } \Gamma, \tag{5}$$

$$Lu_2^{(2k+1)} = f \quad \text{in } \Omega_2, \qquad u_2^{(2k+1)} = \alpha u_2^{(2k)} + (1-\alpha) u_1^{(2k)} \quad \text{on } \Gamma, \tag{6}$$

$$Lu_1^{(2k+2)} = f \quad \text{in } \Omega_1, \qquad \frac{\partial u_1^{(2k+2)}}{\partial v^1} = \beta \frac{\partial u_1^{(2k+1)}}{\partial v^1} + (1-\beta) \frac{\partial u_2^{(2k+1)}}{\partial v^1} \quad \text{on } \Gamma, \tag{7}$$

$$Lu_2^{(2k+2)} = f \quad \text{in } \Omega_2, \qquad \frac{\partial u_2^{(2k+2)}}{\partial v^2} = \beta \frac{\partial u_2^{(2k+1)}}{\partial v^2} + (1-\beta) \frac{\partial u_1^{(2k+1)}}{\partial v^2} \quad \text{on } \Gamma, \tag{8}$$

where $\alpha, \beta \in (0, 1)$ are relaxation parameters that will be determined to ensure and/or accelerate the convergence of the iterative procedure. Since these parameters obviously depend on the domain partition and the original PDE problem it is in general rather difficult to estimate their optimum values. However, theoretical analysis and numerical experiments will indicate that it is reasonable to simply choose $\alpha = \beta = \frac{1}{2}$ for many computations.

As easily seen in this iterative scheme, we impose continuity of the pressure variable u and the flux variable $\partial u / \partial v$ on the interface alternately at each iteration level. When the iteration converges, the limit of the sequence $u_n^{(k)}$ should be the solution of the original problem.

3. Convergence analysis of the differential problem

In this section, we analyze the convergence of the proposed domain decomposition method for PDE problems on rectangular subdomains. We consider the model problem given by

$$-\Delta u + \gamma u = f \quad \text{in } \Omega \equiv [-x_1, x_2] \times [-1, 1], \qquad u = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where $x_1, x_2 > 0$ and γ is a positive constant. We then split the domain Ω into the two subdomains $\Omega_1 \equiv [-x_1, 0] \times [-1, 1]$ and $\Omega_2 \equiv [0, x_2] \times [-1, 1]$, so that the interface line Γ is at $x = 0$. If we denote by $u_i^{(j)}$ the solution of the domain decomposition method on subdomain Ω_i at iteration j, then it is easy to see that the corresponding error functions $e_i^{(j)}$ defined by $e_i^{(j)}(x, y) \equiv u(x, y) - u_i^{(j)}(x, y)$ for

$(x,y)\in\Omega_i$ satisfy, for $k=0,1,\ldots,$ the PDE problems

$$\begin{array}{ll|ll}
-\Delta e_1^{(2k+1)}+\gamma e_1^{(2k+1)}=0 & \text{in } \Omega_1, & -\Delta e_2^{(2k+1)}+\gamma e_2^{(2k+1)}=0 & \text{in } \Omega_2,\\
e_1^{(2k+1)}=r(\alpha,e_1^{(2k)},e_2^{(2k)}) & \text{on } \Gamma, & e_2^{(2k+1)}=r(\alpha,e_2^{(2k)},e_1^{(2k)}) & \text{on } \Gamma,\\
e_1^{(2k+1)}=0 & \text{on } \partial\Omega_1\backslash\Gamma, & e_2^{(2k+1)}=0 & \text{on } \partial\Omega_2\backslash\Gamma,
\end{array} \tag{10}$$

and

$$\begin{array}{ll|ll}
-\Delta e_1^{(2k+2)}+\gamma e_1^{(2k+2)}=0 & \text{in } \Omega_1, & -\Delta e_2^{(2k+2)}+\gamma e_2^{(2k+2)}=0 & \text{in } \Omega_2,\\
\dfrac{\partial e_1^{(2k+2)}}{\partial x}=r\left(\beta,\dfrac{\partial e_1^{(2k+1)}}{\partial x},\dfrac{\partial e_2^{(2k+1)}}{\partial x}\right) & \text{on } \Gamma, & \dfrac{\partial e_2^{(2k+2)}}{\partial x}=r\left(\beta,\dfrac{\partial e_2^{(2k+1)}}{\partial x},\dfrac{\partial e_1^{(2k+1)}}{\partial x}\right) & \text{on } \Gamma,\\
e_1^{(2k+2)}=0 & \text{on } \partial\Omega_1\backslash\Gamma, & e_2^{(2k+2)}=0 & \text{on } \partial\Omega_2\backslash\Gamma,
\end{array} \tag{11}$$

where $r(c,w,z)\equiv cw+(1-c)z$ is a convex combination of w and z.

We next set $\gamma_i=\gamma+(\frac{1}{2}\mathrm{i}\pi)^2$ and define the functions

$$\begin{aligned}
&\Psi_i(y)=\sin(\tfrac{1}{2}\mathrm{i}\pi(y+1)),\\
&\Phi_i(x)=\frac{\sinh(\sqrt{\gamma_i}(x+x_1))}{\sinh(\sqrt{\gamma_i}x_1)} \quad\text{and}\quad Z_i(x)=\frac{\sinh(\sqrt{\gamma_i}(x_2-x))}{\sinh(\sqrt{\gamma_i}x_2)}.
\end{aligned} \tag{12}$$

It is easy to see that these functions are, respectively, solutions of the following problems:

$$\Psi_i''(y)+(\tfrac{1}{2}\mathrm{i}\pi)^2\Psi_i(y)=0 \quad \text{for } y\in(-1,1), \qquad \Psi_i(-1)=\Psi_i(1)=0, \tag{13}$$

$$-\Phi_i''(x)+\gamma_i\Phi_i(x)=0 \quad \text{for } x\in(-x_1,0), \qquad \Phi_i(-x_1)=0, \qquad \Phi_i(0)=1 \tag{14}$$

and

$$-Z_i''(x)+\gamma_i Z_i(x)=0 \quad \text{for } x\in(0,x_2), \qquad Z_i(0)=1, \qquad Z_i(x_2)=0. \tag{15}$$

We can now expand, at iteration j, the error functions in each subdomain in terms of the functions given in (12) as follows:

$$e_1^{(j)}(x,y)=\sum_{i=1}^{\infty}a_i^{(j)}\Phi_i(x)\Psi_i(y), \qquad e_2^{(j)}(x,y)=\sum_{i=1}^{\infty}b_i^{(j)}Z_i(x)\Psi_i(y). \tag{16}$$

The coefficients of the above series are precisely the coefficients of the expansions of the functions $e_i^{(2k+1)}$ and $(\partial e_i^{(2k+2)})/\partial x$ on Γ and are given, for $k=0,1,2,\ldots,$ by

$$a_i^{(2k+1)}=\int_{-1}^{1}[\alpha e_1^{(2k)}(0,y)+(1-\alpha)e_2^{(2k)}(0,y)]\Psi_i(y)\,\mathrm{d}y, \tag{17}$$

$$b_i^{(2k+1)}=\int_{-1}^{1}[\alpha e_2^{(2k)}(0,y)+(1-\alpha)e_1^{(2k)}(0,y)]\Psi_i(y)\,\mathrm{d}y, \tag{18}$$

$$a_i^{(2k+2)} = t_i(x_1)\int_{-1}^{1}\left[\beta\frac{\partial e_1^{(2k+1)}(0,y)}{\partial x} + (1-\beta)\frac{\partial e_2^{(2k+1)}(0,y)}{\partial x}\right]\Psi_i(y)\,\mathrm{d}y, \tag{19}$$

and

$$b_i^{(2k+2)} = -t_i(x_2)\int_{-1}^{1}\left[\beta\frac{\partial e_2^{(2k+1)}(0,y)}{\partial x} + (1-\beta)\frac{\partial e_1^{(2k+1)}(0,y)}{\partial x}\right]\Psi_i(y)\,\mathrm{d}y, \tag{20}$$

where $t_i(x) = [\tanh(\sqrt{\gamma_i}x)]/\sqrt{\gamma_i}$. Using the orthogonality of the Ψ_i's in $L^2(\Gamma)$, Eqs. (19) and (20) become

$$a_i^{(2k+2)} = t_i(x_1)[\beta a_i^{(2k+1)}\Phi_i'(0) + (1-\beta)b_i^{(2k+1)}Z_i'(0)] \tag{21}$$

and

$$b_i^{(2k+2)} = -t_i(x_2)[\beta b_i^{(2k+1)}Z_i'(0) + (1-\beta)a_i^{(2k+1)}\Phi_i'(0)], \tag{22}$$

respectively. Adding the squares of the above two equalities, using the fact that

$$\Phi_i'(0) = \sqrt{\gamma_i}\coth(\sqrt{\gamma_i}x_1), \qquad Z_i'(0) = -\sqrt{\gamma_i}\coth(\sqrt{\gamma_i}x_2),$$

and setting $\rho_i = \tanh(\sqrt{\gamma_i}x_2)/\tanh(\sqrt{\gamma_i}x_1)$, we have that

$$[a_i^{(2k+2)}]^2 + [b_i^{(2k+2)}]^2 = [\beta a_i^{(2k+1)} - (1-\beta)b_i^{(2k+1)}\rho_i^{-1}]^2 + [-\beta b_i^{(2k+1)} + (1-\beta)a_i^{(2k+1)}\rho_i]^2. \tag{23}$$

Similarly, relations (17) and (18) give

$$\begin{aligned} a_i^{(2k+1)} &= \alpha a_i^{(2k)}\Psi_i(0) + (1-\alpha)b_i^{(2k)}Z_i(0) = \alpha a_i^{(2k)} + (1-\alpha)b_i^{(2k)}, \\ b_i^{(2k+1)} &= \alpha b_i^{(2k)}Z_i(0) + (1-\alpha)a_i^{(2k)}\Psi_i(0) = \alpha b_i^{(2k)} + (1-\alpha)a_i^{(2k)}. \end{aligned} \tag{24}$$

Substituting Eqs. (24) into Eq. (23) we see that

$$\begin{aligned} (a_i^{(2k+2)})^2 + (b_i^{(2k+2)})^2 = {} & \{\alpha^2[\beta^2 + (1-\beta)^2\rho_i^2] + (1-\alpha)^2[\beta^2 + (1-\beta)^2\rho_i^{-2}] \\ & - 2\alpha\beta(1-\alpha)(1-\beta)(\rho_i + \rho_i^{-1})\}(a_i^{(2k)})^2 \\ & + \{\alpha^2[\beta^2 + (1-\beta)^2\rho_i^{-2}] + (1-\alpha)^2[\beta^2 + (1-\beta)^2\rho_i^2] \\ & - 2\alpha\beta(1-\alpha)(1-\beta)(\rho_i + \rho_i^{-1})\}(b_i^{(2k)})^2 \\ & + \{2\alpha(1-\alpha)[2\beta^2 + (1-\beta)^2(\rho_i^2 + \rho_i^{-2})] \\ & - 2\beta(1-\beta)(1 - 2\alpha + 2\alpha^2)(\rho_i + \rho_i^{-1})\}a_i^{(2k)}b_i^{(2k)}. \end{aligned} \tag{25}$$

To determine the optimum values for the relaxation parameters α and β one needs to minimize the above expression with respect to these parameters. We were unable to solve this two-parameter minimization problem and therefore we set $\alpha = \frac{1}{2}$. This choice is well justified from the analysis of the one-dimensional case which is not presented here. We proceed by giving next, without proof, a simple lemma of calculus.

Lemma 1. *Let* $\rho_i = \tanh(\sqrt{\gamma_i}x_2)/\tanh(\sqrt{\gamma_i}x_1)$, *then when* $x_1 > x_2$ *we have* $x_2/x_1 < \rho_1 \leqslant \rho_i < 1$, *and when* $x_1 < x_2$ *we have* $1 < \rho_i \leqslant \rho_1 < x_2/x_1$.

Notice that $\|e_1^{(2k+1)}|_\Gamma\|^2 = \sum_{i=1}^{\infty}(a_i^{(2k+1)})^2$ and $\|e_2^{(2k+1)}|_\Gamma\|^2 = \sum_{i=1}^{\infty}(b_i^{(2k+1)})^2$, where $\|\cdot\|$ denotes the norm of the space $L^2(\Gamma)$. Next we give our main convergence result.

Theorem 2. *Set* $s = (x_1 + x_2)/\min\{x_1, x_2\}$. *If* $\alpha = \frac{1}{2}$ *then for the model problem and its decomposition* (4), *we have that*:

(i) *The sequences* $u_1^{(j)}$, $u_2^{(j)}$ *converge if* $\beta \in (0, \frac{2}{3}]$ *and if*

$$(1-\beta)^2 s^2 - 2\beta(1-\beta)s - 2(1-2\beta) < 2.$$

(ii) *The optimum value of the relaxation parameter* β *is given by* $\beta_{\text{opt}} = (s^2 + s - 2)/(s^2 + 2s)$ *and the following error relation holds*:

$$\sum_{j=1}^{2} \|e_j^{(2k+2)}|_\Gamma\|^2 \leqslant \frac{s-2}{2s} \sum_{j=1}^{2} \|e_j^{(2k)}|_\Gamma\|^2. \tag{26}$$

Proof. With $\alpha = \frac{1}{2}$, Eq. (25) becomes

$$(a_i^{(2k+2)})^2 + (b_i^{(2k+2)})^2 = \tfrac{1}{4}g(\rho_i)[a_i^{(2k)} + b_i^{(2k)}]^2 \leqslant \tfrac{1}{2}g(\rho_i)[(a_i^{(2k)})^2 + (b_i^{(2k)})^2],$$

where $g(\rho_i) = 2\beta^2 + (1-\beta)^2(\rho_i^2 + \rho_i^{-2}) - 2\beta(1-\beta)(\rho_i + \rho_i^{-1})$. Setting $\sigma_i = \rho_i + \rho_i^{-1}$ and applying Lemma 1 we have that $2 \leqslant \sigma_i \leqslant (x_1 + x_2)/\min\{x_1, x_2\} = s$.

Define now the function

$$\bar{g}(\sigma) := (1-\beta)^2\sigma^2 - 2\beta(1-\beta)\sigma + 4\beta - 2$$

and differentiate it to get

$$\bar{g}'(\sigma) = 2(1-\beta)[(1-\beta)\sigma - \beta] \geqslant 0, \quad \text{for } \beta \leqslant \frac{\sigma}{1+\sigma}.$$

When $\beta \leqslant \frac{2}{3}$, we have that $\bar{g}'(\sigma_i) \geqslant 0$, $\forall i$ and

$$(a_i^{2(k+1)})^2 + (b_i^{2(k+1)})^2 \leqslant \tfrac{1}{2}\bar{g}(s)[(a_i^{(2k)})^2 + (b_i^{(2k)})^2],$$

which implies part (i) of the theorem.

The function $\bar{g}(s)$ achieves its minimum with respect to β when $\beta = (s^2 + s - 2)/(s^2 + 2s)$, and $\min_\beta(\bar{g}(s)) = (s-2)/2s$. □

To illustrate the rapid convergence of the method we examine the size of the error reduction factor in (26) for some typical decompositions of Ω.

Corollary 1. *With* $\alpha = \frac{1}{2}$, *this domain decomposition method enjoys the following properties*:

(i) *For* $x_1 = x_2$ *and* $\beta_{\text{opt}} = \frac{1}{2}$, *the method converges after two iterations* (*one Dirichlet sweep and one Neumann sweep*).

(ii) *For* $x_1 = 2x_2$ (*or* $x_2 = 2x_1$), *and* $\beta_{\text{opt}} = \frac{2}{3}$, *we have the following error reduction formula.*

$$\sum_{j=1}^{2} \|e_j^{(2k+2)}|_\Gamma\|^2 \leqslant \frac{1}{6}\sum_{j=1}^{2} \|e_j^{(2k)}|_\Gamma\|^2. \tag{27}$$

(iii) *For* $x_1 = \frac{3}{2}x_2$ (*or* $x_2 = \frac{3}{2}x_1$), *and* $\beta_{\text{opt}} = \frac{6.35}{11.25}$, *we have*

$$\sum_{j=1}^{2} \|e_j^{(2k+2)}|_\Gamma\|^2 \leqslant \frac{1}{10}\sum_{j=1}^{2} \|e_j^{(2k)}|_\Gamma\|^2. \tag{28}$$

It is already known that the shapes of the subdomains play an important role in the convergence of several domain decomposition methods. As the next corollary exhibits, this happens also for our method.

Corollary 2. *When* $\alpha = \beta = \frac{1}{2}$, *the proposed method converges if either* $(2 - \sqrt{3})x_1 \leqslant x_2 \leqslant x_1$, *or* $x_1 \leqslant x_2 \leqslant (2 + \sqrt{3})x_1$.

Proof. Let $\alpha = \beta = \frac{1}{2}$. Then Eq. (25) implies that

$$\sum_{j=1}^{2} \|e_j^{(2k+2)}|_\Gamma\|^2 \leqslant \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4}(\rho_i^2 + \rho_i^{-2}) - \frac{1}{2}(\rho_i + \rho_i^{-1})\right]\sum_{j=1}^{2} \|e_j^{(2k)}|_\Gamma\|^2.$$

Convergence is achieved if $\frac{1}{4}[1 + \frac{1}{2}(\rho_i^2 + \rho_i^{-2}) - (\rho_i + \rho_i^{-1})] < 1$, which is equivalent to

$$2 - \sqrt{3} < \frac{\tanh(\sqrt{\gamma_i}x_2)}{\tanh(\sqrt{\gamma_i}x_1)} < 2 + \sqrt{3}.$$

In view of Lemma 1, the proof is now complete. □

As Theorem 2 shows, this domain decomposition method always converges if suitable relaxation parameters are chosen; using the optimal relaxation parameters, the error reduction factor is less than $\frac{1}{2}$ which indicates the fast convergence of the method. Furthermore, Corollary 1 gives some typical examples illustrating the possibility of fast convergence while Corollary 2 indicates that the method with trivial relaxation parameters still converges for a wide range of domain decompositions.

4. Numerical experiments

In this section, numerical examples are presented to confirm the theoretical results given above. A systematic performance evaluation of the proposed and similar domain decomposition methods is under way [4] and will be presented elsewhere. Here we show that, at least in certain cases of practical importance, the method proposed here outperforms the methods proposed by Funaro et al. [1], and Lions [2]. For this we have implemented these three methods using the ELLPACK software system [3]. It is important to point out here that we have observed by experimenting in the ELLPACK framework that the usage of different PDE discretization methods (various Galerkin, finite difference and collocation) does not affect the convergence properties of the method. In all computations below, we apply second-order finite difference discretizations on uniform grids in each of the two subdomains. The resulting linear systems of algebraic equations are solved by banded Gaussian elimination. Single precision is used for all calculations. The initial guesses are always taken to be zero. The errors are evaluated in the L^∞ norm over the two subdomains at each iteration.

We select the following model problems defined on $\Omega \equiv [0,1]\times[0,1]$.

Example 1. $-\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = f$, in Ω, $u=g$, on $\partial\Omega$. The functions f and g are chosen such that the exact solution is $u(x,y) = \sin(\frac{\pi}{2}x)y(1-y)$.

Example 2. $-\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 + 0.5u = f$, in Ω, $u=0$, on $\partial\Omega$. The function f is chosen such that the exact solution is $u(x,y) = 3e^{x+y}x(1-x)y(1-y)$.

We denote the domain decomposition methods as follows:

Method 1. The method described in this paper with $\alpha=\beta=\frac{1}{2}$.

Method 2. The method proposed by Funaro et al. [1] with $\theta=\frac{1}{2}$.

Method 3. The method proposed by Lions [2] with $\lambda=1$.

Tables 1 and 3 contain the errors in the L^∞ norm in the case of equal sized subdomains, while Tables 2 and 4 contain the errors in the L^∞ norm in the case of unequal sized subdomains. The

Table 1
Numerical results for Example 1 with interface at $x=0.5$. The errors are shown in the L^∞ norm

	Grid size $\frac{1}{30}\times\frac{1}{30}$			Grid size $\frac{1}{60}\times\frac{1}{60}$		
Iteration	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
1	$1.76\cdot 10^{-1}$	$1.76\cdot 10^{-1}$	$1.92\cdot 10^{-1}$	$1.76\cdot 10^{-1}$	$1.76\cdot 10^{-1}$	$1.91\cdot 10^{-1}$
2	$1.19\cdot 10^{-4}$	$4.34\cdot 10^{-4}$	$1.88\cdot 10^{-1}$	$4.39\cdot 10^{-5}$	$1.39\cdot 10^{-4}$	$1.90\cdot 10^{-1}$
3	$5.29\cdot 10^{-5}$	$5.73\cdot 10^{-5}$	$1.87\cdot 10^{-1}$	$1.22\cdot 10^{-5}$	$2.73\cdot 10^{-5}$	$1.90\cdot 10^{-1}$

Table 2
Numerical results for Example 1 with interface at $x=0.65$. The errors are shown in the L^∞ norm

	Grid size $\frac{1}{30}\times\frac{1}{30}$			Grid size $\frac{1}{60}\times\frac{1}{60}$		
Iteration	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
1	$1.12\cdot 10^{-1}$	$2.12\cdot 10^{-1}$	$1.45\cdot 10^{-1}$	$2.13\cdot 10^{-1}$	$2.13\cdot 10^{-1}$	$1.45\cdot 10^{-1}$
2	$1.29\cdot 10^{-2}$	$1.93\cdot 10^{-2}$	$1.43\cdot 10^{-1}$	$1.28\cdot 10^{-2}$	$1.90\cdot 10^{-2}$	$1.45\cdot 10^{-1}$
3	$1.31\cdot 10^{-3}$	$1.73\cdot 10^{-3}$	$1.42\cdot 10^{-1}$	$1.36\cdot 10^{-3}$	$1.66\cdot 10^{-3}$	$1.45\cdot 10^{-1}$

Table 3
Numerical results for Example 2 with interface at $x = 0.5$. The errors are shown in the L^∞ norm

	Grid size $\frac{1}{30} \times \frac{1}{30}$			Grid size $\frac{1}{60} \times \frac{1}{60}$		
Iteration	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
1	$5.41 \cdot 10^{-1}$	$5.41 \cdot 10^{-1}$	$4.28 \cdot 10^{-1}$	$5.41 \cdot 10^{-1}$	$5.41 \cdot 10^{-1}$	$4.07 \cdot 10^{-1}$
2	$2.26 \cdot 10^{-3}$	$3.17 \cdot 10^{-3}$	$4.26 \cdot 10^{-1}$	$6.67 \cdot 10^{-4}$	$9.62 \cdot 10^{-4}$	$3.72 \cdot 10^{-1}$
3	$3.14 \cdot 10^{-4}$	$1.25 \cdot 10^{-3}$	$4.24 \cdot 10^{-1}$	$1.12 \cdot 10^{-4}$	$3.85 \cdot 10^{-4}$	$3.38 \cdot 10^{-1}$
4	$1.25 \cdot 10^{-4}$	$1.25 \cdot 10^{-3}$	$4.24 \cdot 10^{-1}$	$1.12 \cdot 10^{-4}$	$3.85 \cdot 10^{-4}$	$3.38 \cdot 10^{-1}$

Table 4
Numerical results for Example 2 with interface at $x = 0.35$. The errors are shown in the L^∞ norm

	Grid size $\frac{1}{30} \times \frac{1}{30}$			Grid size $\frac{1}{60} \times \frac{1}{60}$		
Iteration	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
1	$4.42 \cdot 10^{-1}$	$4.98 \cdot 10^{-1}$	$1.28 \cdot 10^{-1}$	$4.24 \cdot 10^{-1}$	$5.01 \cdot 10^{-1}$	$1.14 \cdot 10^{-1}$
2	$3.95 \cdot 10^{-2}$	$4.88 \cdot 10^{-2}$	$1.27 \cdot 10^{-1}$	$4.08 \cdot 10^{-2}$	$4.92 \cdot 10^{-2}$	$9.12 \cdot 10^{-2}$
3	$3.00 \cdot 10^{-3}$	$4.70 \cdot 10^{-3}$	$1.25 \cdot 10^{-1}$	$3.31 \cdot 10^{-3}$	$4.55 \cdot 10^{-3}$	$6.90 \cdot 10^{-2}$
4	$2.06 \cdot 10^{-3}$	$1.31 \cdot 10^{-3}$	$1.25 \cdot 10^{-1}$	$8.07 \cdot 10^{-4}$	$7.21 \cdot 10^{-4}$	$6.90 \cdot 10^{-2}$

numerical results from Tables 1–4 show that our method performs quite well, being slightly better than Method 2 [1] and much better than Method 3 [2]. Note that the subdomain problems at each iteration level in our method are completely independent and thus the computation is easily parallelizable, while the subdomain problems at each iteration level in Method 2 [1] cannot be fully parallelized in such a natural way.

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